REFERENCES

1. SULIKASHVILI R.S., On the effect of third- and fourth-order moments of inertia on the motion of a rigid body. PMM 51, 2, 1987.

Translated by L.K.

PMM U.S.S.R.,Vol.51,No.5,pp.597-600,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

EQUATIONS OF MOTION OF A CARRIER SUPPORTING DYNAMICALLY UNBALANCED AND ASYMMETRIC FLYWHEELS IN AN INERTIAL MEDIUM*

V.A. KONOPLEV

The methods described in /1-5/ are used to derive the equations of motion of a body supporting dynamically unbalanced and asymmetric flywheels in an inertial fluid. The equations combine the accuracy of inclusion of inertial effects with the compactness of matrix notation, with the convenience of constructing the computational procedures based on modern matrix processing facilities of the digital computer without resorting to the scalar equations. The equations obtained are used to formulate a problem of programmed rotation of flywheels, ensuring that the carrier moves as required, provided it exists.

The equations obtained can be used for a straightforward investigation of the motion of a vibrating table under the condition that the position, the inertial characteristics and the modes of motion are all known, and for determining the above characteristics which ensure that the table moves in a prescribed manner (the control problem).

We shall use, for simplicity, a single symbol $E_k = (O_k, [e^k])$ for all rigid bodies of the system, and for the associated Cartesian systems coordinate with the origin O_k and an orthonormed basis, $[e^k] = (e_1^k, e_2^k, e_3^k)$, $e_1^k = \| 1, 0, 0 \|^T$, $e_2^k = \| 0, 1, 0 \|^T$, $e_3^k = \| 0, 0, 1 \|^T$, so that E_0 will denote the inertial coordinate system, E_1 is the body of the carrier, E_p $(p = 2, 3, \ldots, n)$ the instruments under test installed on E_1, E_s $(s = 2, 3, \ldots, m)$ are the flywheels, including those which may be mounted on the instruments under test.

The dynamic screw of such a system is described in E_0 in the form

$$Z_0^0 = \mathbf{L}_1^{00} Z_1^1, \quad \mathbf{L}_1^{00}: E_0 \to E_1 \tag{1}$$

$$Z_1^1 = \mathbf{K}_1^1 \dot{V}_1^{01} + \sum_{s=1}^{m} \mathbf{L}_s^{11} \mathbf{\Theta}_s^s f_s \varphi_s^{\cdot} \tag{2}$$

$$\mathbf{K}_1^1 = \mathbf{\Theta}_1^1 + \sum_{p=2}^{n} \mathbf{L}_p^{11} \mathbf{\Theta}_p^p \mathbf{L}_p^{11, T} + \sum_{s=2}^{m} \mathbf{L}_s^{11} \mathbf{\Theta}_s^s \mathbf{L}_s^{11, T} + \mathbf{\Lambda}_{11}^{11} + \mathbf{\Lambda}_{p1}^{p1}$$

Here Z_1^1 is the same screw in E_1; $\mathbf{L}_1^{00} = \mathbf{T}_1^{00} [C_1^0]$ is a (6×6)-matrix situated in E_0 /1/, and

$$\mathbf{T}_1^{00} = \begin{Vmatrix} E & 0 \\ \langle O_1^0 \rangle^0 & E \end{Vmatrix}, \quad [C_1^0] = \begin{Vmatrix} C_1^0 & 0 \\ 0 & C_1^0 \end{Vmatrix} \tag{3}$$

where $\langle O_1^0 \rangle^0$ is a skew symmetric (3×3) matrix generated by the position vector O_1^{00} of O_1 in E_0 on the basis $[e^0]$; E is a unit (3×3) matrix, $C_1^0 = C_3(\psi_1) C_2(\theta_1) C_1(\varphi_1)$ is a (3×3) matrix of the orientation $[e^1]$ on $[e^0]$, $([e^1] = [e^0] C_1^0)$ is the simplest (3×3) matrix of

*Prikl.Matem.Mekhan.,51,5,763-766,1987

rotation by the angle α with unit vector e_i^1 $(i = 1, 2, 3)$, and $q = \|O_1^{00}, \psi_1, \theta_1, \varphi_1\|^T \in R_6$ is the vector of generalized coordinates of E_1. We have adopted above a "ship-like" sequence of rotations. When using the "aircraft" or Eulerian angles, we should adopt $C_1^0 = C_2(\psi_1) C_3(\theta_1) C_1(\varphi_1)$ or $C_1^0 = C_3(\psi_{11}) \cdot C_1(\varphi_1) C_3(\psi_{12})$ respectively /1, 6/.

The following notation is used in (2): $\Theta_1^1, \Theta_p^p, \Theta_s^s$ are the (6×6) Mises inertia matrices, E_1, E_p and E_s [1]; Λ_{11}^{11}. Λ_{p1}^{p1} are the (6×6) matrices of attached masses E_1 and E_p, are computed in E_1 [2]; $\mathbf{L}_k^{11}$: $E_1 \to E_k$ $(k = p, s)$ as in (3), and the quantities ψ_s, θ_s in the matrices $C_s^1 = C_3(\psi_s) C_2(\theta_s) C_1(\varphi_s)$ are constant angles of orientation of the flywheels E_s in $[e^1]$, φ_s, are the angles of rotation of the flywheels, O_s^{11} are position vectors of the flywheels E_s in E_1 on the basis $[e^1]$; $V_1^{01} = \| v_1^{01}, \omega_1^{01} \|^T$ is the vector of quasivelocities of E_1; $f_s = \| 0, e_1^s \|^T \in R_6$, e_1^s is the unit vector of the axis of rotation of the flywheels E_s by the angle φ_s.

The left lower $\langle r_c^s \rangle^s m_s$ and right upper $\langle r_c^s \rangle^{s,T} m_s$ (3×3) block of matrices Θ_s^s define the part of inertial effects determined by the dynamic imbalance of the flywheels ($r_c^{ss} \in E_s$ is the radius vector of the centre of mass of the flywheel in E_s and the basis $[e^s]$), the second part of the effects is determined by the dynamic asymmetry of the flywheels ($\theta_{22}^s \neq \theta_{33}^s$ are the moments of inertia in the matrix θ_s^s of the inertia tensor represented by the lower (3×3) block of the matrix Θ_s^s). Using the theorem on the change in kinetic screw of the system and taking into account (1), we obtain /1/

$$Z_1^{1*} + \Phi_1^{01} Z_1^1 = F_1^1 \tag{4}$$

$$Z_1^{1*} = \mathbf{K}_1^1 V_1^{01*} + \mathbf{K}_1^{1*} V_1^{01} + \sum_{s=2}^{m} \mathbf{L}_s^{11} \Theta_s^s f_s \varphi_s^{\cdot\cdot} + \sum_{s=2}^{m} \mathbf{L}_s^{11} [\langle e_1^s \rangle] \Theta_s^s f_s \varphi_s. \tag{5}$$

$$\mathbf{K}_1^{1*} = \sum_{s=2}^{m} \mathbf{L}_s^{11} ([\langle e_1^s \rangle] \Theta_s^s - \Theta_s^s [\langle e_1^s \rangle]) \mathbf{L}_s^{11,T} \varphi_s^{\cdot} \tag{6}$$

$$F_1^1 = P_1^1 + \sum_{p=2}^{n} \mathbf{L}_p^{11} P_p^p + \sum_{s=2}^{m} \mathbf{L}_s^{11} P_s^s \tag{7}$$

Here $[\langle e_1^s \rangle]$ is a block diagonal (6×6) matrix with the blocks $\langle e_1^s \rangle$) on the principal diagonal; P_k^k $(k = 1, p, s)$ is the sum of the dynamic screws of the external forces acting on E_k and E_k (the weight, Archimedes, aerodynamic, hydrodynamic, etc.) /2/; Φ_1^{01} is a (6×6) matrix with (3×6) rows of the form $\| \langle \omega_1^0 \rangle^1 \mid 0 \|$, $\| \langle v_1^0 \rangle^1 \mid \langle \omega_1^0 \rangle^1 \|$; $*$ is the symbol denoting a derivative in the E_1 system of coordinates attached to the body.

In deriving relations (4)-(6) we have used the kinematic equation

$$\mathbf{L}_s^{11*} = \mathbf{L}_s^{11} \Phi_s^{1s} = \mathbf{L}_s^{11} [\langle e_1^s \rangle] \varphi_s$$

Substituting relations (5) and (6) into (4) and carrying out the necessary reduction, we obtain the equation of motion in the inertial fluid of the body supporting the dynamically unbalanced and asymmetric flywheels in quasivelocities

$$\mathbf{K}_1^1 V_1^{01*} + \mathbf{M}_1^1 V_1^{01} + \sum_{s=2}^{m} (\mathbf{I}_s \varphi_s^{\cdot\cdot} + \mathbf{J}_s \varphi_s^{\cdot}) = F_1^1 \tag{8}$$

Here

$$\mathbf{M}_1^1 = \Phi_1^{01} \mathbf{K}_1^1 + \mathbf{K}_1^{1*}, \quad \mathbf{I}_s = \mathbf{L}_s^{11} \Theta_s^s f_s \tag{9}$$
$$\mathbf{J}_s = (\Phi_1^{01} \mathbf{L}_s^{11} + \mathbf{L}_s^{11} [\langle e_1^s \rangle] \varphi_s^{\cdot}) \Theta_s^s f_s$$

Using the equations of kinematics /2, 3, 5/

$$V_1^{01} = \mathbf{A}_1^{01} q_1^{0\cdot}, \quad V_1^{01} = \mathbf{A}_1^{01} q_1^{0\cdot\cdot} + \mathbf{D}_1^{01} q_1^{0\cdot} \tag{10}$$

we transform Eq.(8) to the form

$$\mathbf{I}_1^0 q_1^{0\cdot\cdot} + \mathbf{J}_1^0 q_1^{0\cdot} + \mathbf{I} q^{\cdot\cdot} + J q^{\cdot} = F_1^1 \tag{11}$$
$$\mathbf{I}_1^0 = \mathbf{K}_1^1 \mathbf{A}_1^{01}, \quad \mathbf{J}_1^0 = \mathbf{K}_1^1 \mathbf{D}_1^{01} + \mathbf{M}_1^1 \mathbf{A}_1^{01}$$
$$\mathbf{I} = \| \mathbf{I}_2 | \mathbf{I}_3 | \dots | \mathbf{I}_m \|$$
$$\mathbf{J} = \| \mathbf{J}_2 | \mathbf{J}_3 | \dots | \mathbf{J}_m \|$$
$$q^{\cdot} = \| \varphi_2^{\cdot}, \varphi_3^{\cdot}, \dots, \varphi_m^{\cdot} \|^T, \quad q^{\cdot\cdot} = \| \varphi_2^{\cdot\cdot}, \varphi_3^{\cdot\cdot}, \dots, \varphi_m^{\cdot\cdot} \|^T$$

Thus the presence on the carrier of dynamically unbalanced and asymmetric flywheels leads to the following changes in the equations of motion of a free rigid body /1/: the Mises matrix Θ_1^1 of the carrier is increased by the sum of the matrices Θ_s^s generated by the operators $\mathbf{L}_s^{11}$ in E_{11} (the second relation in (2)); an analogous effect in the term $\Phi_1^{01}\mathbf{K}_1^1$ of the matrix M_1^1 /9/;

the appearance in the matrix M_1^1 of the matrix K_1^{1*} which takes into account the inertial effects connected with the dynamic inbalance and asymmetry of the flywheels (6); the appearance of terms of the type $Iq^{\cdot\cdot}$, $Jq^{\cdot}$. The presence of an inertial medium leads to an increase in the inertia matrix of the system K_1^1 (2) by the matrix of attached masses of the body and the instruments.

Analysing the above changes in Eqs.(8) and (11) we find, that the motion of the carrier in the inertial medium, where the flywheels are dynamically unbalanced, differs essentially from the motion of the usual gyrostat (without taking the above factors into account /7, 8/). Having chosen in the latter case ($r_c^{ss} \equiv 0, \Lambda_{11}^{11} = 0$) the central axes of inertia of the whole system as E_1, we obtain two (3×3) matrix equations modelling the independent translational and rotational motions of the gyrostat. It is usually the latter that is of practical interest /7, 8/.

When the medium is inertial and the flywheels are dynamically unbalanced, the translational and rotational motions of the carrier are inertially inseparable from each other, irrespective of the choice of the attached coordinate system. The presence of the last two matrices in the matrix K_1^1 (2) and of the matrix K_1^{1*} (6) in the matrix M_1^1 (9), implies that motion along any of the generalized coordinates q_1^0 for an arbitrarily short time generates a motion along the remaining coordinates of this vector. The effect remains in force in special cases (an inertial medium only, or dynamic inbalance of the flywheels only). We note that all terms shown in (8) and (11) have an oscillatory character (through the matrix L_3), with variable frequency depending on the frequency of rotation of the flywheels.

Using Eq.(11) we can obtain a mathematical formulation of the problem of controlling the oscillatory motion of the carrier caused by the inertial effects of the dynamically unbalanced rotating flywheels.

Let us rewrite Eq.(11) in the form

$$Iq^{\cdot\cdot} + Jq^{\cdot} = P(t), \quad P(t) = F_1^1 - I_1^0 q_1^{0\cdot\cdot}(t) - J_1^0 q_1^{0\cdot}(t) \tag{12}$$

where $q_1^0(t)$ is the given motion of the carrier.

If $\dim q < 6$ (the number of flywheels is < 6), then the problem has no solution (except in some special cases of the motion $q_1^0(t)$). If $\dim q = 6$, then the problem has a unique solution where $\det I \neq 0$.

Let $\dim q = l > 6$. In this case the problem of determining the required law of variation of the vector $q(t)$ reduces to the classical theory of optimal control in which the methods of solution are well-known. Let us divide the vector $q^{\cdot}$ into two parts, one containing six independent coordinates $q_+ \in R_6$, and the other ($l-6$) independent coordinates $q_- \in R_{l-6}$, $q = \| q_-, q_+\|^T$. We choose, as the control vector, the vector of acceleration of the independent coordinates (which can be manipulated as required) $u = q_-^{\cdot\cdot}$. In this case the problem has the following form when $\det I_- \neq 0$:

$$\begin{aligned} &x^{\cdot} = f(x, u, t), \quad x(t_0) = x_0 \\ &x = \|q, q^{\cdot}\|^T, \quad J(x, u) \to \min \\ &u = q_-^{\cdot\cdot}, \quad x(t) \in E_x, \quad u(t) \in E_u \\ &f(x, u, t) = \| q^{\cdot}, q^{\cdot\cdot} \|^T = \| q^{\cdot}, q_-^{\cdot\cdot}, \; q_+^{\cdot\cdot} \|^T = \\ &\quad \| q^{\cdot}, u, I_+^{-1}(P(t) - I_- u - Jq^{\cdot}) \|^T \end{aligned} \tag{13}$$

Here I_+, I_- are the parts of the matrix I corresponding to the parts $q_+^{\cdot\cdot}$, $q_-^{\cdot\cdot}$ of the vector $q^{\cdot\cdot}$; E_x, E_u are the domains of admissible values of the vectors x, u; $J(x, u)$ is a functional.

We must remember that (8) and (11) are the equations of motion not of the system, but of the body of the carrier. Therefore, the solution of problem (13) yields only a programmed motion of the flywheels, which can be used to design the force control modules (FCM) of the system.

The real motion of the system acted upon by the FCM according to the above program can be studied using additional appropriately chosen equations from /2, 3/.

REFERENCES

1. KONOPLEV V.A., Matrix forms of the equations of motion of a free rigid body. Izv. Akad. Nauk SSSR, MTT, 6, 1985.
2. KONOPLEV V.A., Dynamics of the controlled motion of an elastic manipulator in an inertial agitated fluid. Izv. Akad. Nauk SSSR, MTT, 4, 1986.
3. KONOPLEV V.A. Matrix equations of motion of an elastic kinematic chain with rotating masses attached to the links. Prikl. Mekhanika, 22, 7, 1986.
4. KONOPLEV V.A., The use of matrix methods to investigate the kinematics of the complex motion of a body. Prikl. mekhanika, 20, 9, 1984.

5. KONOPLEV V.A. Matrix methods in the kinematics of the complex motion of a rigid body and a system of rigid bodies. Sbornik nauchno-metoicheskikh statei po teoreticheskoi mekhanike. Moscow, Vyssh. Shk., 17, 1986.
6. LUR'YE A.I., Analytical Mechanics. Moscow, Fizmatgiz, 1961.
7. WITTENBURG J., Dynamics of Systems of Rigid Bodies. Stuttgart, B.G. Teubner, 1977.
8. LITVIN-SEDOI M.Z., Mechanics of systems of connected rigid bodies. Itogi nauki i tekhniki. Ser. Obshchaya mekhanika, Moscow, VINITI, 5, 1982.
9. SARYCHEV V.A., Problems of the Orientation of Artificial Satellites. Itogi nauki i tekhniki. Ser. Issledovaniye kosmicheskogo prostranstva. 11, Moscow, VINITI, 1978.

Translated by L.K.

PMM U.S.S.R.,Vol.51,No.5,pp.600-606,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

AN ANISOTROPICALLY ELASTIC SPHERE IN FREE MOTION*

V.V. NOVIKOV

Corrections are found to the inertial tensor components of a rotating sphere in the case of small anisotropy of its elastic properties of general form. Singularities in the behaviour of the freely rotating sphere due to the intrinsic elasticity are discussed in specific examples. Without making any assumptions on the smallness of the anisotropy, the strain is calculated for a sphere having a plane of isotropy. It was shown /1, 2/ in the problem of the motion of a solid deformable body around a centre of mass under the simplifying assumptions that the natural vibration frequencies greatly exceed the angular velocity while the internal friction forces ensure sufficiently rapid damping of the natural vibrations, that taking account of the intrinsic elasticity of the body is equivalent to the action of a moment on it proportional to the fourth power of the angular velocity component and calculated by means of the solution of the quasistatic problem of the deformation of a rotating body.

The moment corresponding to the influence of the intrinsic elasticity has been calculated /3/ for a body close in shape to a sphere. The homogeneous anisotropically elastic sphere in free motion considered in this paper is still another example when the solution of this problem can be obtained by analytic means. The representation of the behaviour of this system can turn out to be useful when considering questions of the earth's motion in connection with the hypothesis that the earth has the features of a complex crystal.

Let us represent the stress tensor in the form

$$\sigma_{ij}=\lambda u_{ll}\delta_{ij}+2\mu u_{ij}+c_{ijnm}u_{nm}$$

where u_{ij} is the strain tensor, λ and μ are Lamé constants, and u_i are the components of the displacement vector. The tensor of the elastic constants c_{ijnm} satisfies the following symmetry conditions /5/:

$$c_{ijnm}=c_{jinm}=c_{ijmn}=c_{nmij}$$

and has 21 indepdent components in the most general case of an anisotropic linearly elastic body.

We shall consider only the almost Eulerian motions of a deformable body. This is possible if it is sufficiently rigid and the vibrations of the elastic body that occur damp out rapidly /2/. The elastic constants are such that the following inequalities are satisfied

$$\varepsilon\ll\delta\ll 1\ \ (\varepsilon=\rho R^2/(\mu t_*^2)) \tag{1}$$

where ρ is the density, R is the radius of the sphere, t_* is the characteristic time of sphere motion as a whole, and δ is the ratio of the greatest of the elastic constants c_{ijnm} to

*Prikl.Matem.Mekhan.,51,5,767-774,1987